Eur. Phys. J. B **52**, 249–253 (2006)
DOI: 10.1140/epjb/e2006-00288-4

THE EUROPEAN PHYSICAL JOURNAL B

Stochastic resonance: influence of a $f^{-\kappa}$ noise spectrum

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Received 6 February 2006 / Received in final form 29 March 2006
Published online 31 July 2006 –

Abstract. With the aim of studying *stochastic resonance* (SR) in a double-well potential when the noise source has a spectral density of the form $f^{-\kappa}$ (with varying κ), we have extended a procedure introduced by Kaulakys et al. (Phys. Rev. E **70**, 020101 (2004)). In order to achieve an analytical understanding of the results, we have obtained an effective Markovian approximation that allows us to make a systematic study of the effect of such noise on the SR phenomenon. A comparison of the numerical and analytical results shows an excellent qualitative agreement indicating that the effective Markovian approximation is able to correctly describe the general trends.

PACS. 02.50.Ey Stochastic processes – 05.40.-a Fluctuation phenomena, random processes, noise, and Brownian motion – 02.50.-r Probability theory, stochastic processes, and statistics

1 Introduction

Stochastic resonance (SR) is one of the most interesting *noise-induced phenomena*, that arises from the interplay between *deterministic* and *random* dynamics in a *nonlinear* system [1]. This phenomenon has been largely studied for more than two decades due to its great interest not only from a basic point of view but also for its technological interest and biological implications [1,2].

Most of those studies have used white or colored noises, with a few exceptions where wider classes of noise were considered. For instance, in [3] SR in systems subject to a colored and non Gaussian noise was studied. Due to the ubiquity of $f^{-\kappa}$ noise in a large variety of physical, biological and even economical phenomena [4], it is apparent that there is interest in the effect of such a particular characteristic of the noise's power spectral density (PSD) on the response of a system subject to it. This study is the objective of the present study. However, there are several studies closely related with the present one, as indicated by the following examples. In [5], the authors studied, through numerical simulations, the behavior of the signal-to-noise ratio (SNR) gain in a level crossing detector and a Schmidt trigger, when subject to a colored noise composed of a periodic train pulse plus a Gaussian $f^{-\kappa}$ noise with variable κ. Their results indicate that the maximum of the SNR is larger for white noise, and moves towards large noise intensities for increasing κ. In [6] experimental evidence was found that noise can enhance the homeostatic function in the human blood pressure regulatory system. Related to this, other experimental evidence was found in [7] that an externally applied f^{-1} noise, added to the usual white noise, contributes to sensitizing the baroflex function in the human brain. In [8], a model of traffic junction with a main and a side road, it was found that the effect of a Gaussian $f^{-\kappa}$ noise with $\kappa \geq 0$ showed an overall traffic efficiency enhancement.

It has been experimentally demonstrated [9] that an SR-like effect can be obtained in rat sensory neurons with white, f^{-1} and f^{-2} noise, and that, under particular conditions, f^{-1} noise can be better than white noise to enhance a neuron's response. Indeed, it was shown that it is possible to enhance the SR effect in a FitzHugh-Nagumo model submitted to a colored noise with $f^{-\kappa}$ [10], and that the optimal noise variance of SR could be minimized with $\kappa \approx 1$ [11].

To our knowledge, there has been no theoretical study on the connection of the PSD noise characteristic and the enhancement of the stochastic resonance effect. The lack of a theory, and the work of Kaulakys and collaborators [12], who have introduced a method to generate f^{-1} noise over a wide range of frequencies (see also [13,14]), has motivated us to discuss how to extend such a procedure for positive and negative values of the stochastic variable. Furthermore, we exploit this procedure to analyze *analytically* the effect of a noise spectrum of the form $f^{-\kappa}$ with varying κ, on the SR phenomenon in a simple double-well potential.

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In the following Section we present the model system to be studied and the procedure to generate the $f^{-\kappa}$ noise. Afterwards we discuss an effective Markovian approximation, and exploit it to study the SR phenomenon. Finally we discuss the results and draw some general conclusions.

2 The system

2.1 Stochastic differential equations

The starting point of our analysis is the following system of stochastic differential equations

$$\dot{x} = f(x) + g(x)y(t) \tag{1}$$

$$\dot{y} = \frac{u(y)}{\tau} + \frac{D}{\tau}v(y)\xi(t). \tag{2}$$

The first equation describes the evolution of the coordinate x of a particle diffusing in a double well potential $U_0(x)$, given by $U_0(x) = -\int^x f(\zeta)d\zeta = \frac{x^4}{4} - \frac{x^2}{2}$. This particle is subject to a noise $y(t)$ through a multiplicative constraint given by the general function $g(x)$, to be defined later.

The second equation corresponds to the Langevin equation driving the noise $y(t)$, inspired by the work of Kaulakys et al. [12]. We note in passing that when $u(y) = -y$ and $v(y) = 1$, $y(t)$ reduces to the well known Ornstein-Uhlenbeck process. The choice of this particular form for the evolution of y is due to our intention to preserve the properties of the PSD discussed in [12]. In this last equation we consider a new potential $V(y) = -\int^y u(s)ds$, and a (white) noise $\xi(t)$ that enters in a multiplicative form with a function $v(y)$. The strength of this multiplicative noise will be proportional to the parameter D, while the time evolution of the complete equation is characterized by the parameter τ.

Next, we describe in detail the non linear function previously introduced in equations (1, 2). We consider the following form for the function $u(y)$

$$u(y) = \alpha y^3 - \beta y^5 + s(y)y^4 \tag{3}$$

where $s(y)$ indicates the sign of y (i.e, -1 if $y < 0$ and $+1$ if $y \geq 0$). For the function $v(y)$ we adopt

$$v(y) = |y|^\mu + c, \tag{4}$$

where both the exponent μ and the constant c are positive (>0).

The above indicated forms change the symmetry of the potential $V(y)$ and, in addition, when compared with the work in [12], increases the range of the noise variable from $[0, +\infty)$ to $(-\infty, +\infty)$, as is shown in Figure 1. The parameter c allows the random variable y to adopt negative values when $c > |y|^\mu$.

2.2 Characteristic of the noise variable y

The most relevant aspect of the process y is its power spectral density with a $1/f$ frequency behavior. Kaulakys et al. [12] have shown that when $c = 0$ and $\mu = 5/2$, the noise y exhibits a $1/f$ functionality over a wide range of frequencies. For $c > 0$, but small, this property is still valid, as we show in Figures 2 and 3.

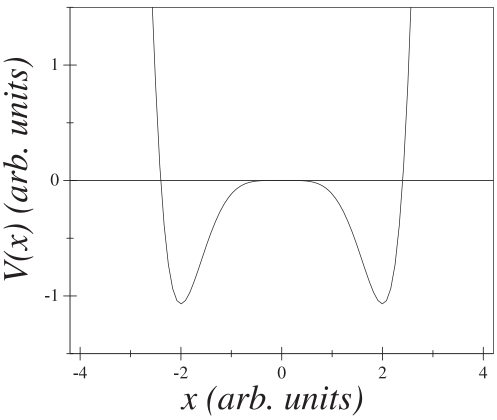

Fig. 1. Symmetric potential $V(y)$ as derived from equation (3). We have used the following values: $\alpha = 5 \times 10^{-4}$ and $\beta = \frac{1}{2}$.

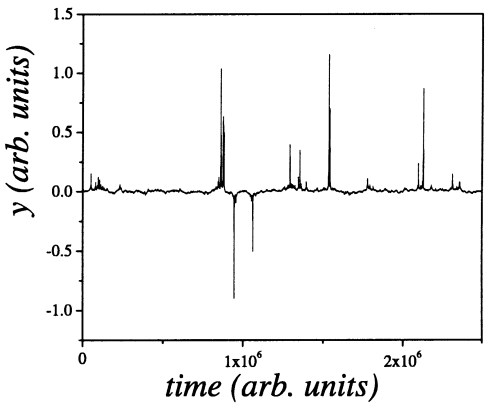

Fig. 2. A typical realization for equation (2) using $\mu = 5/2$ and $c = 1 \times 10^{-4}$. The PSD of this realization shows a $1/f$ behavior, see Figure 3.

When the exponent μ changes, the PSD behaves as $1/f^k$, with $k < 1$ for $\mu < 5/2$. We will use this property to evaluate the mean-first-passage-time (MFPT) and, afterwards, the signal-to-noise ratio (SNR). In particular we have used $\mu = 3/2$, yielding $k \simeq 3/4$.

3 Effective Markovian theory

In order to be able to obtain some analytical results, we resort here to an *effective Markovian approximation*, that allows us to reduce, in the variable's space, the original non Markovian problem to a Markovian one. The ultimate goal of this procedure is to achieve a consistent single variable Fokker-Planck approximation for the probability distribution of the original variable. This analysis is analogous to the so called *unified colored noise approximation* (UCNA) (introduced in [15,16]), which consists of an adiabatic elimination-like procedure. Exploiting such an approach we are able to find an effective Markovian

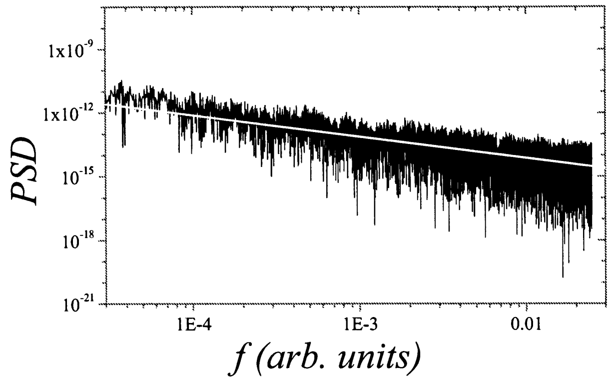

Fig. 3. PSD for the variable y, as indicated in equation (2). We used the same values of parameters as in Figure 2. The white line corresponds to a linear fitting, resulting in a slope $\kappa = -1.004 \pm 0.005$.

Fokker-Planck equation (FPE) for the probability density $P(x,t)$. The procedure is the following (the prime will indicate derivation respect to the variable x).

3.1 Adiabatic procedure

Deriving equation (1) respect to the time we have

$$\ddot{x} = f'(x)\dot{x} + g'(x)\dot{x}y + g(x)\dot{y}. \tag{5}$$

Now, assuming an adiabatic behavior, we eliminate $\ddot{x}$, and using equation (2) we obtain

$$0 \simeq f'(x)\dot{x} + g'(x)\dot{x}\left[\frac{\dot{x} - f(x)}{g(x)}\right] + g(x)\left[\frac{u(Z(x))}{\tau} + \frac{Dv(Z(x))\xi(t)}{\tau}\right], \tag{6}$$

where we have defined $Z(x) = Z_0(x) + Z_1(x)$, with $Z_0(x) = -\frac{f(x)}{g(x)}$ and $Z_1(x) = \frac{\dot{x}}{g(x)}$. Now, in order to obtain a useful effective Markovian description, we need to resort to further approximations, as follows

$$u(Z(x)) \approx u(Z_0(x)) + u'(Z_0(x))\, Z_1(x) \tag{7}$$

and similarly for v

$$v(Z) \approx v(Z_0) + v'(Z_0)\, Z_1(x). \tag{8}$$

Adopting now $g(x) = 1$, that implies $Z_0 \equiv -f(x)$ and $Z_1 \equiv \dot{x}$, we have

$$0 = f'(x)\dot{x} + \frac{u(Z_0(x))}{\tau}\dot{x} + D\frac{v(Z_0(x))}{\tau}\xi(t) + O(\dot{x}\,\xi(t)). \tag{9}$$

From the above, the effective equation for the process x adopts the following form

$$\dot{x} = -\frac{u(Z_0(x)) + Dv(Z_0(x))\xi(t)}{\tau f'(x) + u'(Z_0(x))} = A_1(x) + B_1(x)\xi(t), \tag{10}$$

where we can write the limits for $A(x)$ and $B(x)$ when $\tau \to 0$ as

$$A_1(x) \to \frac{Z_0(x)}{3} = A(x), \tag{11}$$

$$B_1(x) \to -\frac{Dv(Z_0(x))}{u'(Z_0(x))} = B(x). \tag{12}$$

Finally, using the above indicated approximations, the stochastic differential equation for the process x reads

$$\dot{x} = A(x) + B(x)\,\xi(t), \tag{13}$$

with $A(x)$ and $B(x)$ defined above.

3.2 Fokker-Planck equation

The FPE associated with the Langevin equation, equation (13), is

$$\frac{\partial}{\partial t}P(x,t) = -\frac{\partial}{\partial x}[A(x)P(x,t)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[B^2(x)P(x,t)], \tag{14}$$

where the Ito prescription was used [17]. As is well known, the stationary probability distribution (pdf) for this FPE is given by [17]

$$P^{st}(x) = \frac{N}{B(x)}\exp\{-\Phi(x)\}, \tag{15}$$

where N is a normalization factor, and

$$\Phi(x) = 2\int^x \frac{A(\zeta)}{B(\zeta)^2}d\zeta. \tag{16}$$

3.3 Mean-First-Passage-Time and SR

The indicated FPE and its associated pdf allow us to obtain the mean-first-passage-time (MFPT) through a Kramers-like approximation. Using known expressions we obtain for the MFPT [17]

$$T(x_0) = 2\int_a^{x_0}\frac{dy}{\Psi(y)}\int_{-\infty}^{y}\frac{dz\,\Psi(z)}{B(z)^2}, \tag{17}$$

where

$$\Psi(x) = \exp\left\{2\int^x \frac{A(\zeta)}{B(\zeta)^2}d\zeta\right\}. \tag{18}$$

In order to study SR, as usual, we introduce an external signal in the form of a term rocking the double well potential: $U(x) = U_0(x) + S(t)$, with $S(t) = S_0\sin(\omega t)$ (in what follows we adopt $\omega = 1.33 \times 10^{-5}$). Exploiting the so called "two-state approximation" [1] (see also [18]), we define the SNR as the ratio of the strength of the output signal and the broadband noise output evaluated at the signal frequency ω, obtaining [1]

$$SNR \propto \left\{\frac{1}{T}\frac{dT}{dS}\right\}_{S=0}, \tag{19}$$

where the derivative of the T in the above expression is evaluated, as indicated, at $S = 0$.

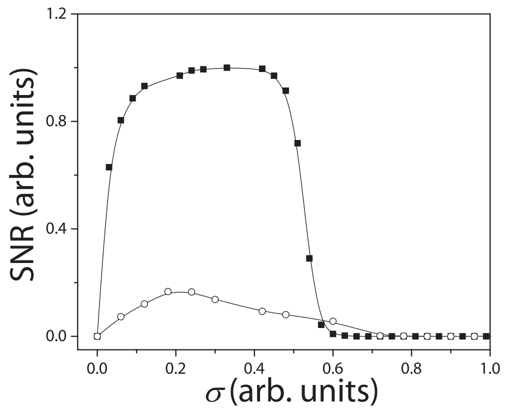

Fig. 4. SNR obtained when simulating the full set of equations (1, 2). Here σ corresponds to the noise intensity defined through the distribution width, as indicated in the text. Squares and circles corresponds for $\mu = 3/2$ and $\mu = 5/2$ respectively. The lines are a guide to the eye only.

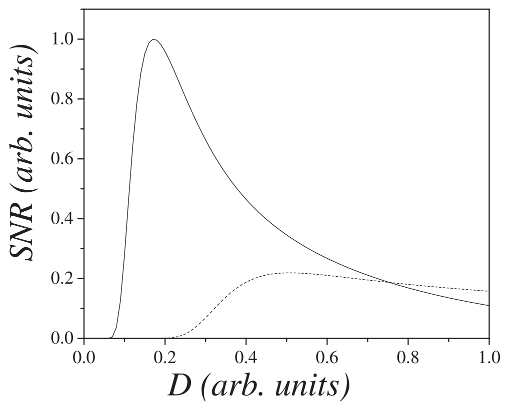

Fig. 5. SNR obtained using equation (19), as derived from the two state theory. Continuous and dashed lines correspond to $\mu = 3/2$ and $\mu = 5/2$ respectively.

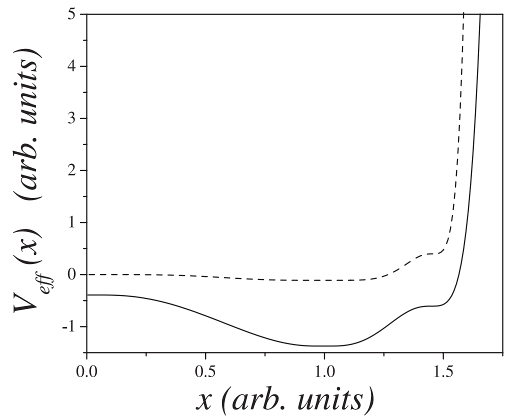

Fig. 6. Effective potential, equation (18), showing the different behavior of the wells for different values of the exponent μ: continuous line $\mu = 5/2$, dotted line $\mu = 3/2$. Due to the symmetry, we only show positive values of x.

4 Results and conclusion

We have carried out extensive numerical simulations of the full set of equations (1) and (2) in order to obtain the SNR. The results are shown in Figure 4. Also, in Figure 5 we show the SNR computed using the effective Markovian theory, obtained through equations (17) and (19). In order to be able to compare the results we have normalized the curves. Also, in order to have a well defined noise intensity σ, the variance of $y(t)$ (as described by the Eq. (2)) was obtained numerically in all the simulation, and directly related to σ. As in Sections 3.2 and 3.3, here we have assumed $g(x) = 1$.

The comparison of Figures 4 and 5 makes it apparent that the results obtained using the effective Markovian theory are in very good (qualitative) agreement with those from simulations. This is in accord with previous results obtained for different systems [3,16]. We can conclude that such an effective Markovian (UCNA-like) approximation, as discussed in Section 3, offers an adequate framework to obtain effective Markovian approximations for a much wider class of systems than the one to which it was originally applied [15].

The above results are in complete agreement with those of [5]. That is; the maximum of the SNR is larger for white noise, and it moves towards large noise intensities for increasing κ. In order to gain some physical insight into this behavior it is worth remarking that the function defined by equation (18) is directly related to an effective potential within the approximation we used

$$V_{eff}(x) \approx D \ln \Psi(x) = D \left\{ 2 \int^x \frac{A(\zeta)}{B(\zeta)^2} d\zeta \right\}. \qquad (20)$$

The behavior of such a potential reveals the consequences of changing the exponent μ: when $\mu = 5/2$ (i.e. the PSD is $1/f$), the effective potential shows a well defined well; but as μ decreases the well is less defined as shown in Figure 6. Such a behavior of the effective potential explains why the SNR increases when μ decreases. The general theory shows that the SNR increase is proportional to the Kramers rate r_K, that is given by

$$r_K = \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{\Delta V_{eff}}{D} \right\}, \qquad (21)$$

where ΔV_{eff} is the height of the barrier in the effective potential separating the attractors. Hence, the reduction of the SNR with increasing κ (or decreasing μ) could be directly related to the marked reduction of the barrier separating the attractors in the effective potential picture. This also explains the reason for the shift of the SNR maximum towards larger values of the noise intensity.

In conclusion, this is a first step towards an analytical understanding of two very important, connected, and ubiquitous, aspects in natural processes. These are the $1/f^k$ behavior of noise's PSD, and its role in signal detection via the SR mechanism. The physical picture provided by the indicated effective Markovian approximation offers an adequate framework to analyze and understand the main qualitative trends of such a phenomenon. Furthermore, we expect to apply the same scheme to other noise induced phenomena when subject to $1/f^k$ noise. This will be the subject of further work.

M.A.F. would like to thank the support of CONICET, Argentina, to the Santa Fe Institute for its support and hospitality, and to M. Ballard for fruitful discussions related to the one-dimension FPE description. H.S.W. wants to thank the European Commission for the award of a *Marie Curie Chair*, and acknowledges partial support from the Ministerio de Educación y Ciencia, Spain, through the project CGL2004-02652/CLI.

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